A NOTE ON THE PICARD NUMBER OF SINGULAR FANO 3-FOLDS

GLORIA DELLA NOCE

Abstract. Using a construction due to C. Casagrande and further developed by the author in [DN12], we prove that the Picard number of a non-smooth Fano 3-fold with isolated factorial canonical singularities, is at most 6.

Introduction

Let X be a Fano 3-fold. If X is smooth, we know from the classifiction results in [MM81], that its Picard number ρ_X is at most 10. Moreover, if $\rho_X \geq 6$, then X is isomorphic to a product $S \times \mathbb{P}^1$, where S is a smooth Del Pezzo surface.

If X is singular, bounds for ρ_X are known only in particular cases. If X is toric and has canonical singularities, then $\rho_X \leq 5$ ([Bat82] and [WW82]). If X has Gorenstein terminal singularities, then $\rho_X \leq 10$, because X has a smoothing which preserves ρ_X (see [Nam97, Thorem 11] and [JR11, Theorem 1.4]). If, instead, X has Gorenstein canonical singularities, it does not admit, in general, a smooth deformation (see [Pro05, Example 1.4] for an example). In this setting, the following holds.

Theorem 0.1. [DN12, Theorem 1.3] *Let X be a 3-dimensional $\mathbb{Q}$-factorial Gorenstein Fano variety with isolated canonical singularities. Then $\rho_X \leq 10$.*

The proof of this theorem uses a construction introduced by C. Casagrande in [Cas12], and relies on the result of [BCHM10] that Fano varieties are *Mori dream spaces* (see [HK00] for the definition).

In this paper, using the same construction, we show that the bound given by Theorem 0.1 can be improved if X is actually singular and its singularities are also factorial. Our result is the following.

Theorem 0.2. *Let X be a non-smooth factorial Fano 3-fold with isolated canonical singularities. Then $\rho_X \leq 6$.*

In the first section of this paper, we recall some preliminary results from [DN12]; the second section contains the proof of Theorem 0.2 and an observation concerning the case $\rho_X = 6$.

Notation and terminology

We work over the field of complex number.

Let X be a normal variety. We call X *Fano* if $-K_X$ has a multiple which is an ample Cartier divisor. We denote by X_{reg} the non-singular locus of X. We say that X is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier, *i.e.* it admits a multiple which is Cartier. We call X *factorial* if all its local rings are UFD; by [Har77, II, Proposition 6.11], this implies that every Weil divisor of

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X is Cartier. We refer the reader to [KM98] for the definition and properties of terminal and canonical singularities. If X has canonical singularities, it is called *Gorenstein* if its canonical divisor K_X is a Cartier divisor.

We denote with $\mathcal{N}_1(X)$ (resp. $\mathcal{N}^1(X)$) the vector space of one-cycles (resp. $\mathbb{Q}$-Cartier divisors) with real coefficients, modulo the relation of numerical equivalence. The dimension of these two real vector spaces is, by definition, the *Picard number of* X, and is denoted by ρ_X. We denote by $[C]$ (resp. $[D]$) the numerical equivalence class of a one-cycle (resp. a $\mathbb{Q}$-Cartier divisor).

Given $[D] \in \mathcal{N}^1(X)$, we set $D^\perp := \{\gamma \in \mathcal{N}_1(X) | D \cdot \gamma = 0\}$, where $\cdot$ denotes the intersection product. We define $\mathrm{NE}(X) \subset \mathcal{N}_1(X)$ as the convex cone generated by classes of effective curves and $\overline{\mathrm{NE}}(X)$ is its closure. An *extremal ray* R of X is a one-dimensional face of $\overline{\mathrm{NE}}(X)$. We denote by $\mathrm{Locus}(R)$ the subset of X given by the union of curves whose class belongs to R.

A *contraction* of X is a projective surjective morphism with connected fibers $\varphi : X \to Y$ onto a projective normal variety Y. It induces a linear map $\varphi_* : \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$ given by the push-forward of one-cycles. We set $\mathrm{NE}(\varphi) := \overline{\mathrm{NE}}(X) \cap \ker(\varphi_*)$. We say that φ is K_X-negative if $K_X \cdot \gamma < 0$ for every $\gamma \in \mathrm{NE}(\varphi)$.

The *exceptional locus* of φ is the locus where φ is not an isomorphism; we denote it by $\mathrm{Exc}(\varphi)$. We say that φ is *of fiber type* if $\dim(X) > \dim(Y)$, otherwise φ is birational. We say that φ is *elementary* if $\dim(\ker(\varphi_*)) = 1$. In this case $\mathrm{NE}(\varphi)$ is an extremal ray of $\overline{NE}(X)$; we say that φ (or $\mathrm{NE}(\varphi)$) is *divisorial* if $\mathrm{Exc}(\varphi)$ is a prime divisor of X and it is *small* if its codimension is greater than 1.

An elementary contraction from a 3-fold X is called *of type* $(2,1)$ if φ is K_X-negative, birational, $\dim(\mathrm{Exc}(\varphi)) = 2$ and $\dim(\varphi(\mathrm{Exc}(\varphi))) = 1$.

If $D \subset X$ is a Weil divisor and $i : D \to X$ is the inclusion map, we set $\mathcal{N}_1(D, X) := i_*\mathcal{N}_1(D) \subseteq \mathcal{N}_1(X)$.

1. Preliminaries

In the following statement, we collect some results from [DN12]. For the reader's convenience, we recall here the main steps of their proof. We refer the reader to [DN12, Theorem 2.2] for the properties of contractions of type $(2,1)$ defined on mildly singular 3-folds.

Lemma 1.1. [DN12, Theorem 1.2 and its proof - Remark 5.2] *Let X be a $\mathbb{Q}$-factorial Gorenstein Fano 3-fold with isolated canonical singularities. Suppose $\rho_X \geq 6$. Then there exist morphisms*

$$\psi : X \to \mathbb{P}^1 \quad \textit{and} \quad \xi : X \to S,$$

where S is a normal surface with $\rho_S = \rho_X - 1$, and the morphism

$$\pi := (\xi, \psi) : X \to S \times \mathbb{P}^1$$

is finite.

Moreover there exist extremal rays $R_0, \dots, R_m$ ($m \geq 3$) in $\mathrm{NE}(X)$ such that:

- *each R_i is of type $(2,1)$;*
- $\mathrm{NE}(\psi) = R_0 + \cdots + R_m$;
- *for $i = 0, \dots, m$, set $E_i = \mathrm{Locus}\, R_i$ and $Q = \mathrm{NE}(\xi)$. Then*

$$\psi(E_i) = \mathbb{P}^1, \quad \mathcal{N}_1(E_i, X) = \mathbb{R}R_i \oplus \mathbb{R}Q \quad \textit{and} \quad Q \subseteq \bigcap_{i=0}^{m} E_i^\perp;$$

- *ψ factors as $X \xrightarrow{\sigma} \tilde{X} \to \mathbb{P}^1$, where σ is birational, $\tilde{X}$ is a Fano 3-fold with canonical isolated singularities, $\mathrm{NE}(\sigma) = R_1 + \cdots + R_s$, with $m \geq s \in \{\rho_X - 2, \rho_X - 3\}$ and $\sigma(E_1), \ldots, \sigma(E_s) \subset \tilde{X}$ are pairwise disjoint.*

Proof. By [DN12, Remark 5.2], the assumption $\rho_X \geq 6$ implies that all the assumptions of [DN12, Theorem 1.2] are satisfied, from which the existence of the finite morphism π. The properties of its projections ψ and ξ follow by their construction, that we briefly recall. All the details can be found in the proof of [DN12, Theorem 1.2].

By [DN12, Proposition 3.5], there exists an extremal ray $R_0 \subset \mathrm{NE}(X)$ of type $(2, 1)$. Set $E_0 = \mathrm{Locus}(R_0)$; we have $\dim \mathcal{N}_1(E_0, X) = 2$. As in [DN12, Lemma 3.1], we may find a Mori program

$$X = X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \dashrightarrow \cdots \dashrightarrow X_{k-1} \overset{\sigma_{k-1}}{\dashrightarrow} X_k \xrightarrow{\varphi} Y \tag{1.1}$$

where $X_1, \ldots, X_k$ are $\mathbb{Q}$-factorial 3-folds with canonical singularities and, for each $i = 0, \ldots, k-1$, there esists a K_{X_i}-negative extremal ray $Q_i \subset \mathrm{NE}(X_i)$ such that σ_i is either its contraction, if Q_i is divisorial, or its flip, if it is small. Moreover, if $(E_0)_i \subset X_i$ is the transform of E_0 and $(E_0)_0 := E_0$, then $(E_0)_i \cdot Q_i > 0$. Finally, φ is a fiber type contraction to a $\mathbb{Q}$-factorial normal variety Y.

Let us set

$$\{i_1, \ldots, i_s\} := \{i \in \{0, \ldots, k-1\} | \operatorname{codim} \mathcal{N}_1(D_{i+1}, X_{i+1}) = \operatorname{codim} \mathcal{N}_1(D_i, X_i) - 1\}.$$

Then, by [DN12, Lemma 3.3], $s \in \{\rho_X - 2, \rho_X - 3\}$ (in particular $s \geq 3$); moreover, for every $j \in \{1, \ldots, s\}$, Q_{i_j} is a divisorial ray, σ_{i_j} is a birational contraction of type $(2, 1)$ and, if $E_j \subset X$ is the transform of the exceptional divisors of the contraction σ_{i_j} as above, then $E_1, \ldots, E_s$ are pairwise disjoint.

Since $s \geq 3$, [DN12, Proposition 3.5] assures that, for each $j = 1, \ldots, s$, there exists an extremal ray $R_j \subset \mathrm{NE}(X)$ of type $(2, 1)$ such that $E_j = \mathrm{Locus}(R_j)$. The divisor $-K_X + E_1 + \cdots + E_s$ comes out to be nef, and its associated contraction $\sigma : X \to \tilde{X}$ verifies

$$\ker(\sigma_*) = \mathbb{R}R_1 + \cdots + \mathbb{R}R_s \quad \text{and} \quad \mathrm{Exc}(\sigma) = E_1 \cup \cdots \cup E_s.$$

It is thus possible to look at σ a part of a Mori program as in (1.1), and to find a fiber type contraction $\varphi : \tilde{X} \to Y$ giving rise to a morphism $\psi := \varphi \circ \sigma : X \to Y$ as in the statement. In particular, we have $\mathrm{NE}(\psi) = R_0 + \cdots + R_m$, where $m \geq s$ and $R_{s+1}, \ldots, R_m$ are extremal rays of type $(2, 1)$. We notice that, since $\dim(X) = 3$, we have $Y \cong \mathbb{P}^1$ by [DN12, Remark 4.2].

The second projection ξ arises as the contraction associated to a certain nef divisor defined as a combination of the prime divisors $E_0, \ldots, E_m$ constructed above (recall that $E_i = \mathrm{Locus}\, R_i$ for $i = 0, \ldots, m$). It is an elementary contraction and the one-dimensional subspace generated by $\mathrm{NE}(\xi)$ belongs to $\mathcal{N}_1(E_i, X)$ for every $i = 0, \ldots, m$. □

2. Theorem 0.2

Proof of Theorem 0.2. Let us prove that, if $\rho_X \geq 7$, then the morphism $\pi : X \to S \times \mathbb{P}^1$ given by Lemma 1.1 is an isomorphism. This will give a contradiction with our assumptions on the singularities of X, since $S \times \mathbb{P}^1$ is smooth or has one-dimensional singular locus.

We are in the setting of Lemma 1.1; let us keep its notations. By [AW97, Corollary 1.9 and Theorem 4.1(2)], the general fiber of ξ is a smooth rational curve, and the other fibers have

at most two irreducible components (that might coincide) whose whose reduced structures are isomorphic to $\mathbb{P}^1$.

Our assumptions imply that S is factorial: if $C \subset S$ is a Weil divisor, its counterimage $D := \xi^{-1}(C) \subset X$ is a Cartier divisor, because X is factorial. Moreover $D \cdot Q = 0$ (where $Q = \mathrm{NE}(\xi)$), because D is disjoint from the general fiber of ξ. Then $D = \xi^*(C')$ for a certain Cartier divisor C' on S. But then $C = C'$ is Cartier.

Fix $i = 0, \ldots, m$; let $\varphi_i : X \to Y_i$ be the contraction of R_i and set $G_i := \varphi_i(E_i) \subseteq Y_i$, $T_i := \xi(E_i) \subseteq S$:

$$\begin{tikzcd} & E_i \arrow[dl, "\varphi_{i|E_i}"'] \arrow[dr, "\xi_{|E_i}"] & \\ G_i & & T_i. \end{tikzcd}$$

Notice that $T_i \subset S$ is a curve. Indeed, by Lemma 1.1, $E_i \cdot Q = 0$, which implies that $T_i \subset S$ is a (Cartier) divisor and $E_i = \xi^*(T_i)$.

Let f_i be the general fiber of φ_i. Since f_i is a smooth rational curve which dominates T_i, T_i is a (possibly singular) rational curve. The same conclusion holds for G_i, which is dominated by any smooth curve contained in a fiber of ξ over T_i.

We have

$$-1 = E_i \cdot f_i = \xi^*(T_i) \cdot f_i = T_i^2 \cdot \deg(\xi_{|f_i}),$$

from which $-T_i^2 = \deg(\xi_{|f_i}) = 1$. Then the general fiber g of ξ over T_i is a smooth rational curve. Indeed, g has no embedded points, and if, by contradiction, the 1-cycle associated to g is of the type $C_1 + C_2$, then g would intersect f_i in at least two (distinct or coincident) points. This is impossible because g is general and $\deg(\xi_{|f_i}) = 1$.

Then E_i is smooth along the general fibers of both φ_i and ξ; we deduce that E_i is smooth in codimension one. Moreover E_i is a Cohen-Macaulay variety, because X is factorial. Then, by Serre's criterion, E_i is normal. Then the finite morphism $(\xi_{|E_i}, \varphi_{i|E_i}) : E_i \to T_i \times G_i$, which has degree one, factors through the normalization of the target: there is a commutative diagram

$$\begin{tikzcd} E_i \arrow[r, "\tau"] \arrow[dr] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\nu"] \\ & T_i \times G_i. \end{tikzcd}$$

Since τ is finite of degree one, by Zariski Main Theorem, it is an isomorphism. Thus $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\xi_{|E_i} : E_i \to T_i \cong \mathbb{P}^1$ and $\varphi_{i|E_i} : E_i \to G_i \cong \mathbb{P}^1$ are the projections. In particular, since both E_i and T_i are Cartier divisors, they are contained in the smooth loci of, respectively, X and S.

We have

$$(K_X - \xi^*(K_S)) \cdot f_i = (K_{E_i} - \xi^*_{|E_i}(K_{T_i})) \cdot f_i = (\varphi^*_{i|E_i}(K_{G_i})) \cdot f_i = 0.$$

Let F be a general fiber of $\psi : X \to \mathbb{P}^1$. Then F is a smooth Del Pezzo surface and, by Lemma 1.1, $\mathcal{N}_1(F) \subseteq \sum \mathbb{R}[f_i]$; thus $K_X - \xi^*(K_S)$ is numerically trivial in F. Moreover $\zeta := \xi_{|F} : F \to S$ is a finite morphism of degree $d := \deg(\pi)$ and

$$K_F = (K_X)_{|F} = (\xi^*(K_S))_{|F} = \zeta^* K_S; \qquad (2.1)$$

in particular ζ is unramified in the open subset $\xi^{-1}(S_{\mathrm{reg}})$, which contains $E_i \cap F$ for every $i = 0, \dots, m$.

Set $\tilde{F} := \sigma(F) \subset \tilde{X}$, where $\sigma : X \to \tilde{X}$ is the birational contraction given by Lemma 1.1; then $\tilde{F}$ is again a smooth Del Pezzo surface and $\sigma_{|F} : F \to \tilde{F}$ is a contraction. For every $i = 1, \dots, s$, the intersection $E_i \cap F$ is the union of d disjoint curves numerically equivalent to f_i; in particular $\sigma_{|F}$ realizes F as the blow-up of $\tilde{F}$ along $s \cdot d$ distinct points (where $s = \rho_X - \rho_{\tilde{X}}$). Then, recalling that $s \geq \rho_X - 3$ and $\rho_X \geq 7$, we get

$$9 \geq \rho_F = \rho_{\tilde{F}} + s \cdot d \geq 1 + 4d,$$

and then $d \leq 2$. Moreover, if $d = 2$, then $\rho_F = 9$ and, by 2.1,

$$1 = K_F^2 = \zeta^*(K_S) \cdot K_F = 2(K_S)^2,$$

which is impossible because S is factorial and thus K_S^2 is integral. Hence $d = \deg(\zeta) = \deg(\pi) = 1$ and the statement is proved. □

The case $\rho_X = 6$ is more complicated to analyze. Indeed, though Lemma 1.1 still holds in that case, we are not able to conclude that π is an isomorphism and that, as a consequence, X is smooth.

Proposition 2.1. *Let X be a factorial Fano 3-fold with isolated canonical singularities and with $\rho_X = 6$. If X is not smooth, there exists a finite morphism of degree 2*

$$\pi : X \to S \times \mathbb{P}^1,$$

where S is a singular Del Pezzo surface with factorial canonical singularities, $\rho_S = 5$, $(K_S)^2 = 1$. Moreover the ramification locus of π contains a surface R which dominates S.

Proof. We argue as in the proof of Theorem 0.2 and we use the same notations. Since X is not smooth, the degree of π must be 2. Exactly as in the above case, we have

$$K_F = (K_X)_{|F} = (\xi^*(K_S))_{|F} = (\xi^* K_S)_{|F} = \zeta^* K_S, \tag{2.2}$$

and

$$\rho_F = 10 - (K_F)^2 = 10 - 2(K_S)^2, \tag{2.3}$$

so that ρ_F needs to be even. Since $\rho_X = 6$, we have $s \in \{3, 4\}$, and then

$$9 \geq \rho_F = \rho_{\tilde{F}} + 2s.$$

Thus the only possibility is that $\rho_{\tilde{F}} = 2$ and $\rho_F = 8$. By (2.3), we get $(K_S)^2 = 1$.

Let us call R the ramification divisor (possibly trivial) of π. Let C be the general fiber of ξ. Then $C \cong \mathbb{P}^1$ and $\psi_{|C} : \mathbb{P}^1 \to \mathbb{P}^1$ is finite of degree 2. By Hurwitz's formula we have $R \cdot C = 2$, and hence R is not trivial and it dominates S. □

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Dipartimento di Matematica, Università di Pavia, via Ferrata, 1 27100 Pavia - Italy
E-mail address: gloria.dellanoce@unipv.it